SOLUTION OF HEAT-CONDUCTION PROBLEMS BY THE METHOD OF DISCRETE Z-TRANSFORMATION

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The discrete Z-transformation is applied to the solution of heat-conduction problems in a plate with various boundary conditions.

The trend of recent years has been toward a wider use of the apparatus of the discrete Laplace or Z-transformation for the analysis of the dynamics of computer-controlled objects [1-4]. Studies have demonstrated that this apparatus can be successfully applied to those cases where the object of control is a temperature field. In such a case the problem reduces, in practical terms, to the solution of the heat-conduction problem by means of that apparatus. We will consider its application to the example of a temperature field in a plate. For simplicity, we start with the one-dimensional heat-conduction equation

$$\frac{1}{a}\frac{\partial t}{\partial \tau}=\frac{\partial^2 t}{\partial x^2}. \tag{1}$$

We will use the concept of the grid function of time $t[n\Delta\tau, x]$, or $t[n, x]$ in abridged form, the values of which are defined at discrete instants of time $\tau = n\Delta\tau$. The values of this grid function $t[n, x]$ coincide with the values of the continuous function $t(\tau, x)$ at the same instants of time, function $t(\tau, x)$ constituting the principal envelope of the grid function $t[n, x]$. The grid function analog of the first derivative of the continuous function is the first difference, e.g., the reverse difference

$$\Delta t[n, x]=t[n, x]-t[n-1, x]. \tag{2}$$

After a change to referred coordinates, Eq. (1) accordingly becomes

$$f\frac{d^2 t}{du^2}[n,u]-t[n, u]+t[n-1, u]=0. \tag{3}$$

For solving Eq. (3) we will use the discrete Z-transformation realizable by means of the relation

$$Z\{t[n]\}=T(z)=\sum_{n=0}^{\infty} t[n]\, z^{-n}. \tag{4}$$

It is to be noted that multiplication of the transform $T(z)$ by z^{-1} corresponds to a time delay by one discretization interval, viz., $Z^{-1}\{z^{-1}T(z)\} = t[n-1]$, and that, furthermore, the transform of the k-th reverse difference becomes

$$Z\{\Delta^k t[n]\}=(1-z^{-1})^k\, T(z) \tag{5}$$

when the grid function vanishes identically for negative arguments [1].

Application of transformation (4) to Eq. (3) reduces the latter to

$$f\frac{d^2}{du^2}T[z, u]-(1-z^{-1})\,T(z, u)=0. \tag{3a}$$

The solution to Eq. (3a) is

$$T(z, u)=C\cosh\sqrt{f^{-1}(1-z^{-1})}\,u=C\sum_{i=0}^{\infty} P_i(u)\, f^{-i}(1-z^{-1})^i. \tag{6}$$

This solution has been obtained for the boundary condition

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$$\frac{\partial t}{\partial x}[n, 0] = 0 \quad \text{or} \quad \frac{\partial t}{\partial u}[n, 0] = 0 \tag{7}$$

at the surface x = 0 (u = 0). In expression (6) we use the notation

$$P_i(u) = \frac{u^{2i}}{(2i)!}\ . \tag{8}$$

It can be demonstrated that for the boundary condition (7) C = T(z, 0) in solution (6) and therefore

$$T(z, u) = T(z, 0) \sum_{i=0}^{\infty} P_i(u) f^{-i} (1 - z^{-1})^i. \tag{6a}$$

An inverse transformation, according to relation (5), yields

$$t[n, u] = \sum_{i=0}^{\infty} \Delta^i t[n, 0] P_i(u) f^{-i}. \tag{9}$$

Let

$$\Delta^i t[n] = \sum_{\nu=0}^{i} (-1)^{\nu} C_i^{\nu} t[n - \nu],$$

where C_i^{ν} are binomial coefficients (numbers of combinations) and let

$$A_{\nu}(u, f) = (-1)^{\nu} \sum_{i=\nu}^{n} C_i^{\nu} P_i(u) f^{-i}.$$

Then

$$t[n, u] = \sum_{\nu=0}^{n} A_{\nu}(u, f)\, t[n - \nu, 0]. \tag{9a}$$

Expression (9a) written for the transforms becomes

$$T(z, u) = T(z, 0) \sum_{\nu=0}^{n} A_{\nu}(u, f) z^{-\nu}. \tag{9b}$$

We will now consider boundary conditions of the first kind at the heated surface

$$t[n, x = l] = t[n, l]; \quad t = [n, u = 1] = t[n, 1]; \quad Z\{t[n, 1]\} = T(z, 1).$$

With the aid of expression (9b), we find that

$$T(z, u) = T(z, 1) \frac{\sum_{\nu=0}^{n} A_{\nu}(u, f) z^{-\nu}}{\sum_{\nu=0}^{n} A_{\nu}(1, f) z^{-\nu}} \tag{10}$$

or

$$T(z, u) = T(z, 1) \sum_{\nu=0}^{n} B_{\nu}(u, f) z^{-\nu}. \tag{10a}$$

Functions $A_{\nu}(u, f)$ and B(u, f) are related through the recurrence formulas

$$B_i(u) = \frac{1}{A_0(1)} \left[A_i(u) - \sum_{j=1}^{i} A_j(1) B_{i-j}(u) \right]. \tag{11}$$

An inverse transformation in expression (10) can be effected by various methods. We will rewrite this expression as follows:

$$T(z, u) \sum_{\nu=0}^{n} A_{\nu}(1, f) z^{-\nu} = T(z, 1) \sum_{\nu=0}^{n} A_{\nu}(u, f) z^{-\nu}.$$

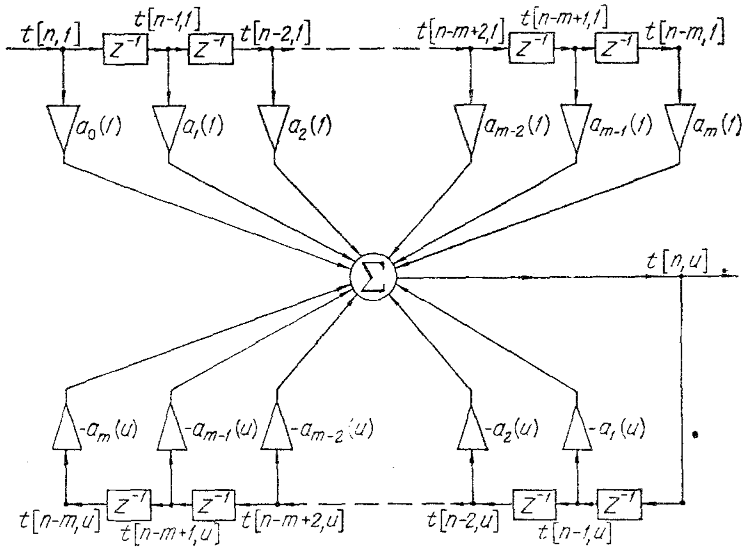

Fig. 1. Schematic block diagram for realization of the transfer function according to expression (10).

Using the originals, we obtain

$$t[n,\ u] = \sum_{0}^{n} a_\nu(u,\ f)\, t[n-\nu,\ 1] - \sum_{1}^{n} a_\nu(1,\ f)\, t[n-\nu,\ u], \tag{12}$$

where

$$a_\nu(u,\ f) = A_\nu(u,\ f)/A_0(1,\ f).$$

Realization of the transfer function corresponding to expression (10) is shown schematically on the block diagram in Fig. 1, based on using adder elements, multiplier elements, and a time delay element. This schematic diagram corresponds to a limitation of the number of terms in the sums in expressions (10) and (12) to $m \leqslant n$. According to the results of analysis, m = 3-5 is sufficient for engineering calculations.

Another realization of the algorithm corresponding to expression (10) is as follows. This expression can be replaced, in accordance with relation (9b), with two expressions for transforms of the temperature functions

$$T(z,\ 1) = T(z,\ 0) \sum_{0}^{m} A_\nu(1,\ f)\, z^{\nu};$$

$$T(z,\ u) = T(z,\ 0) \sum_{0}^{m} A_\nu(u,\ f)\, z^{-\nu}.$$

From these expressions we obtain two expressions for originals of the temperature function

$$t[n,\ u] = \sum_{0}^{m} t[n-\nu,\ 0]\, A_\nu(u,\ f); \tag{12a}$$

$$t[n,\ 0] = \frac{t[n,\ 1]}{A_0(1,\ f)} - \sum_{1}^{m} a_\nu(1)\, t[n-\nu,\ 0]. \tag{12b}$$

The scheme for realization of the algorithm corresponding to expressions (12a) and (12b) is shown in Fig. 2.

From equality (10a) follows the simplest algorithm

$$t[n,\ u] = \sum_{\nu=0}^{m} B_\nu(u,\ f)\, t[n-\nu,\ 1]. \tag{12c}$$

Its realization is shown schematically in Fig. 3.

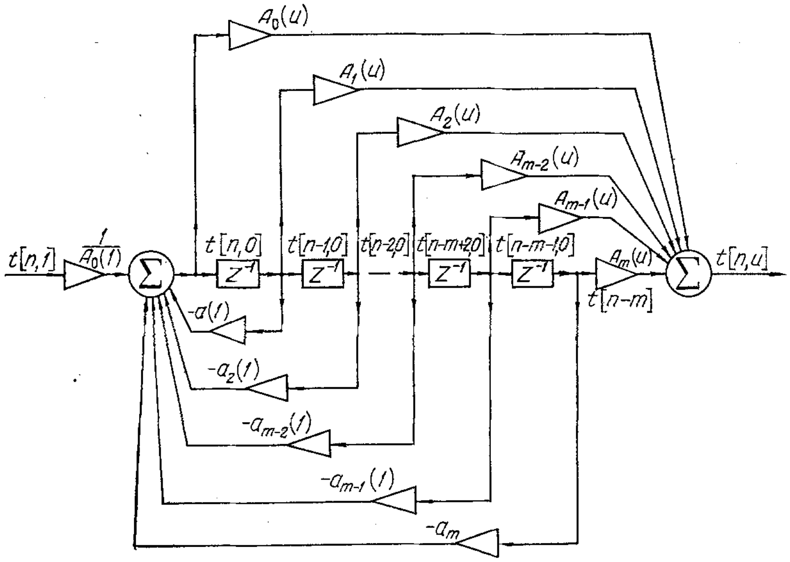

Fig. 2. Schematic block diagram for realization of the transfer function according to expressions (12a) and (12b).

Each of the proposed schemes has its advantages. The scheme in Fig. 1 ensures storage in the memory of the last m values of the sought function, viz., the temperature at the point whose coordinate is u. The scheme in Fig. 2 requires fewer memory cells (time delay elements) for realization and, at the same time, provides immediate determination of two temperatures: at the point whose coordinate is u and at the point on the insulated surface u = 0. The last m values of t(u = 0) are stored in the memory. The scheme in Fig. 3 is simplest of all.

Boundary conditions of the second kind at the heated surface of the plate can be given in terms of the relation

$$\frac{\partial t}{\partial u}[n,1] = y^{\mathrm{II}}[n], \tag{13}$$

which, after a Z-transformation and according to relation (9b), yields the solution for the transform

$$T(z, u) = Y^{\mathrm{II}}(z) = \frac{\sum\limits_{\nu=0}^{m} A_\nu(u, f)\, z^{-\nu}}{\sum\limits_{\nu=0}^{m} A_\nu^{\mathrm{II}}(1, f)\, z^{-\nu}}, \tag{14}$$

where

$$A_\nu^{\mathrm{II}}(1, f) = \frac{dA_\nu}{du}(1, f) = (-1)^\nu \sum_{i=\nu}^{m} C_i^\nu \frac{dP_i}{du}(1)\, f^{-i}; \qquad \frac{dP_i}{du}(1) = \frac{1}{(2i-1)!}. \tag{15}$$

The corresponding expressions for the originals are the same as expressions (12) and (11), with t[n, 1] replaced by $y^{\mathrm{II}}[n]$ and $A_\nu(1, f)$ replaced by $A_\nu^{\mathrm{II}}(1, f)$.

In practice, it is often more convenient to use calculation formulas derived by reduction of the problem with boundary conditions of the second kind to the problem with boundary conditions of the first kind. The gist of this method is that one determines first the temperature of the heated surface of the plate at a given instant of time, t[n, 1], and then the sought temperature.

For the derivation of the corresponding calculation formulas we will use expressions (10a) and (13), which yield

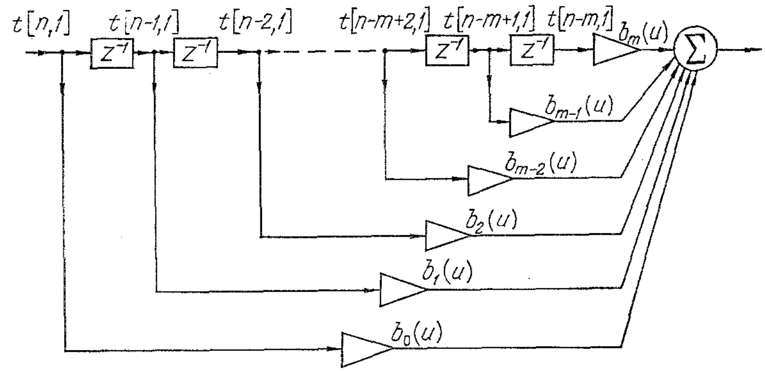

Fig. 3. Schematic block diagram for realization of the algorithm according to expression (12c).

$$Y^{\mathrm{II}}(z)=T(z,\ 1)\sum_{0}^{m}B_{\nu}^{\mathrm{II}}z^{-\nu};\quad B_{\nu}^{\mathrm{II}}=\frac{dB_{\nu}}{du}\ (1,\ f). \tag{16}$$

After a few simple transformations, we obtain the originals

$$t[n,\ 1]=b_0^{\mathrm{II}}y_{\mathrm{II}}[n]-\sum_{\nu=1}^{m}b_{\nu}^{\mathrm{II}}t[n-\nu,\ 1];$$
$$b_0^{\mathrm{II}}=[B_0^{\mathrm{II}}]^{-1};\quad b_{\nu}^{\mathrm{II}}=B_{\nu}^{\mathrm{II}}/B_0^{\mathrm{II}}. \tag{17}$$

Boundary conditions of the third kind at the heated surface can be given, after a change to grid functions, as

$$t[n,\ 1]=t_c[n]-\frac{1}{\mathrm{Bi}[n]}\frac{\partial t}{\partial u}[n,\ 1]. \tag{18}$$

The simplest way to solve the problem when the heat-transfer coefficient is variable is by reduction to the problem with boundary conditions of the first kind. In this case one determines first the temperature t[n, 1] of the heated surface according to the relation

$$t[n,\ 1]=t_c[n]\frac{\mathrm{Bi}[n]}{\mathrm{Bi}[n]+B_0^{\mathrm{II}}}-\frac{B_0^{\mathrm{II}}}{\mathrm{Bi}[n]+B_0^{\mathrm{II}}}\sum_{\nu=1}^{m}b_{\nu}^{\mathrm{II}}t[n-\nu,\ 1], \tag{19}$$

and then the sought temperature according to relations (12). The advantage of this method of calculation is that only one value of Biot number is used on each step, namely the value which corresponds to the given instant of time.

Using these solutions obtained by the method shown here should greatly reduce the volume of calculations in cases where the temperature is to be determined at one or several points across the thickness of a plate (or bodies with other geometrical configurations), since this can be done here without calculating the temperature at other points (as is necessary in conventional grid methods). This feature becomes important in the construction of algorithms for computer-controlled heating of solid objects or equipment components (such as in power plants). Computers used for this purpose have, as a rule, a limited direct-access memory. Meanwhile, the parameters which determine the reliability of the heating (or cooling) process are either the temperatures at specific points of the object (e.g., at the heated surface) or simple functions of these temperatures: rate of change of temperature, mean-integral (over the plate thickness) temperature, temperature difference proportional to thermal stress, etc. The solution obtained here by the methods shown were used by the author in the construction, among others, of algorithms for controlling the heating of large steam turbines during start-up.

NOTATION

t, temperature; x, space coordinate; l, plate thickness; τ, time; Δτ, discretization interval; n and m, integers; $\Delta\tau^*=l^2/a$; $f=\Delta\tau/\Delta\tau^*$; $u=x/\sqrt{a\Delta^*\tau}$; $z=\exp(s\Delta\tau)$; s, Laplace operator; $y_{(\tau)}^{\mathrm{II}}=(l/\lambda)q(\tau)$; λ, thermal conductivity; q(τ), thermal flux; a, thermal diffusivity; $\mathrm{Bi}=\alpha l/\lambda$, Biot number; t_c, temperature of the heating medium; α, heat-transfer coefficient; and T(z), Y(z), transforms of functions t, y obtained by discrete Z-transformation.

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COUPLED PROBLEMS OF MASS TRANSFER BETWEEN SEMI-INFINITELY LARGE REGIONS DURING A CHEMICAL REACTION OF THE SECOND ORDER

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By the method developed in an earlier study [1], an asymptotic expression (at times $t \to \infty$) is derived here for the rate of mass transfer at the boundary between media when in one of them there takes place a chemical reaction of the second order.

We consider the earlier problem [2, 3] concerning mass transfer in a semi-infinitely large region where a chemical reaction of the second order takes place, viz.,

$$\frac{\partial C_1}{\partial t} - D\frac{\partial^2 C_1}{\partial x^2} + kC_1C_2 = 0, \tag{1}$$

$$\frac{\partial C_2}{\partial t} - D\frac{\partial^2 C_2}{\partial x^2} + kC_1C_2 = 0, \tag{2}$$

$$0 \leqslant x < \infty, \quad 0 < t < \infty;$$

$$C_1|_{x=0} = A = \text{const}; \quad \left.\frac{\partial C_2}{\partial x}\right|_{x=0} = 0;$$
$$C_1|_{x=\infty} = 0; \quad C_2|_{x=\infty} = B = \text{const}; \quad C_1|_{t=0} = 0; \quad C_2|_{t=0} = B. \tag{3}$$

The concentration of substance 1 at the boundary is maintained constant. The plane $x = 0$ is impermeable to substance 2.

We introduce for the analysis two new functions

$$S_1 = C_1; \quad S_2 = B - C_2, \tag{4}$$

so that system (1)-(2) can be rewritten as

$$\frac{\partial S_1}{\partial t} - D\frac{\partial^2 S_1}{\partial x^2} + kS_1(B - S_2) = 0,$$
$$\frac{\partial S_2}{\partial t} - D\frac{\partial^2 S_2}{\partial x^2} - kS_1(B - S_2) = 0. \tag{5}$$

Conditions (3) are transformed in an obvious manner. It is essential, for the application of the given method or solution, that $S_2 = 0$ at $t = 0$ and $x = \infty$.

Adding the two equations (5), we obtain

$$\frac{\partial}{\partial t}(S_1 + S_2) - D\frac{\partial^2}{\partial x^2}(S_1 + S_2) = 0, \tag{6}$$

$$(S_1 + S_2)_{x=\infty} = 0; \quad (S_1 + S_2)_{t=0} = 0. \tag{7}$$

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